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0 Fundamental Theorem of Calculus

1. FTC I: If f is a continuous function and a is any constant, consider the function $F(x) = \int_a^x f(t)dt$, which, given x, computes the integral of f from a to x. Then $F'(x) = f(x)$.
   - Example: if $F(x) = \int_2^{x^2} \cos(t)dt$, then $F'(x) = 2x\cos(x^2)$ (don't forget the chain rule!)
2. FTC II: If f is a continuous function and F is an antiderivative of f, then $\int_a^b f(x)dx = F(b) - F(a)$.

The *average value* of a function f over the interval $[a, b]$ is defined as $\frac{1}{b-a}\int_a^b f(x)dx$.

1 Integration Techniques

1.1 u-substitution

u-substitution is "backwards chain rule". The chain rule is $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$, so $\int f'(g(x)) \cdot g'(x)dx = f(g(x))$. So, the idea of u-substitution is as follows: whenever we have some complicated expression we need to integrate involving a *composition of functions* $r(s(x))$, we want to see if we can substitute u for $s(x)$ to simplify the expression (hopefully we can then integrate $r(u)$ easily). To do this, we also need a copy of $s'(x)$ inside the integrand in order to carry out the u-substitution procedure.

Example 1.1. We will evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$. We notice that the numerator has the complicated-looking expression $e^{\sqrt{x}}$, which is a composition of the square root and exponential functions. So we want to substitute $u = \sqrt{x}$. Notice that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, which looks like part of our integrand $\frac{e^{\sqrt{x}}}{\sqrt{x}}$. In more detail, since $du = \frac{1}{2\sqrt{x}}dx$, so $2du = \frac{1}{\sqrt{x}}dx$, we have

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^u(2du) = 2e^u = 2e^{\sqrt{x}}.$$

Don't forget that when you perform definite integration and u-substitute $u = f(x)$, the bounds of integration $\int_a^b$ change to $\int_{f(a)}^{f(b)}$.

1.2 Integration by Parts

Integration by parts is "backwards product rule". The product rule is $(uv)' = uv' + u'v$, so upon integrating both sides, we get $uv = \int uv' + \int u'v$, or $\int uv' = uv - \int u'v$. Therefore, whenever we see a *product of functions* $f(x)g(x)$, we should try an integration by parts.

Recall the LIATE heuristic:

- **L**–logarithmic functions (e.g. $\ln(x+1)$)
- **I**–inverse trigonometric functions (e.g. $\arctan(x), \arccos(2x)$)
- **A**–algebraic functions (e.g. polynomials and power functions, like $2\sqrt{x} + 3x^{-3/2} + 4x^2$)
- **T**–trigonometric functions (e.g. $\sin(x)$)
- **E**–exponential functions (e.g. e^{2x})

A good rule of thumb when choosing u and dv in integration by parts is to choose u to be a function higher up on this list, and dv to be a function farther down. This is because it is much easier to find the antiderivative of A, T, and E functions.

Often times, an integration by parts is used after a u-substitution to make the integrand more tractable.

Example 1.2. We will evaluate $\int \frac{\cos(1/x)}{x^3} dx$. We first notice the complicated-looking expression $\cos(1/x)$; via a substitution $t = \frac{1}{x}$ (so $dt = -\frac{1}{x^2} dx$), the integral becomes $-\int t\cos(t)dt$. This is a product of two elementary functions, so we integrate by parts. Using the LIATE mnemonic. we should choose $u = t$ and $dv = \cos(t)$, so that $du = 1$ and $v = \sin(t)$. So the integral is

$$\int \frac{\cos(1/x)}{x^3} dx = -\int t\cos(t)dt = -\left(t\sin(t) - \int \sin(t)dt\right) = -t\sin(t) - \cos(t) = -\frac{\sin(1/x)}{x} - \cos(1/x).$$

A common trick is "integration by parts with 1"; that is, when integrating a function $f(x)$ with no immediate antiderivative, we apply integration by parts to $1 \cdot f(x)$ with $u = f(x)$ and $v = 1$.

Example 1.3. We will evaluate $\int \arctan(x)dx$. We do not know the antiderivative of arctan, and there are no compositions or products of functions in the integrand, so we introduce a product, since we do know the derivative of arctan. With $\arctan(x) = 1 \cdot \arctan(x)$ and $u = \arctan(x)$, $dv = 1$, $du = \frac{1}{1+x^2}$, $v = x$, we get

$$\int \arctan(x)dx = x\arctan(x) - \int \frac{x}{1+x^2} dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2),$$

where the final integral is done with a u-substitution $u = 1 + x^2$.

There is another common technique of doing integration by parts multiple times until we get an algebraic relation between the original integral and whatever we get from integration by parts. This techniques works well with functions whose derivatives repeat periodically, like e^x or $\sin(x)$.

Example 1.4. We will evaluate $I = \int e^x \sin(x)dx$. One integration by parts with $u = \sin(x)$, $dv = e^x$ gives

$$I = e^x \sin(x) - \int e^x \cos(x)dx.$$

Another integration by parts with $u = \cos(x)$, $dv = e^x$ gives

$$e^x \sin(x) - \int e^x \cos(x) = e^x \sin(x) - \left(e^x \cos(x) - \int e^x(-\sin(x))dx\right) = e^x \sin(x) - e^x \cos(x) - I.$$

So $I = e^x \sin(x) - e^x \cos(x) - I$, meaning that our original integral I is $\frac{e^x \sin(x) - e^x \cos(x)}{2}$.

1.3 Trigonometric Integrals

The key to trig integrals is using trig identities to make the right substitution. We start with the basic identities $\sin^2(x) + \cos^2(x) = 1$, $\tan^2(x) + 1 = \sec^2(x)$. We repeatedly apply these identities until a u-substitution or simplification is possible.

Example 1.5. Here is a basic example:

$$\int \sin^5(x)\cos^3(x)dx = \int \sin^5(x)\cos^2(x)\cos(x)dx = \int \sin^5(x)(1-\sin^2(x))\cos(x)dx = \int u^5(1-u^2)du$$

where we substitute $u = \sin(x)$. The last integral is easily evaluated. Notice that we "leave out a copy of cosine" for the u-substitution to work.

For these types of problems, you should also know double-angle and half-angle identities for sine and cosine.

When we perform trig substitution, there are three main types of substitutions to make, depending on the type of term appearing in the integrand. Below, assume $a > 0$:

- The term looks like $\sqrt{a - x^2}$. In this case, thinking about $1 - \sin^2(x) = \cos^2(x)$, we should try $x = \sqrt{a}\sin(\theta)$ (or $x = \sqrt{a}\cos(\theta)$, depending on the other terms present in the integrand), so that $\sqrt{a - x^2} = \sqrt{a - a\sin^2(\theta)} = \sqrt{a}\sqrt{1 - \sin^2(\theta)} = \sqrt{a}\cos(\theta)$, and this is now hopefully easier to integrate after the substitution.

- The term looks like $\sqrt{x^2 - a}$. In this case, thinking about $\sec^2(x) - 1 = \tan^2(x)$, we should try $x = \sqrt{a}\sec(\theta)$, so that $\sqrt{x^2 - a} = \sqrt{a\sec^2(\theta) - a} = \sqrt{a}\tan(\theta)$, which should be easier to integrate.

- The term looks like $\sqrt{x^2+a}$. In this case, thinking about $\tan^2(x)+1=\sec^2(x)$, we should try $x=\sqrt{a}\tan(\theta)$, so that $\sqrt{x^2-a}=\sqrt{a\tan^2(\theta)+a}=\sqrt{a}\sec(\theta)$, which should be easier to integrate.

Integrals that are good to remember:

$$\int \tan(x)dx = -\ln|\cos(x)|, \quad \int \sec(x)dx = \ln|\sec(x)+\tan(x)|, \quad \int \frac{1}{x^2+a}dx = \frac{1}{\sqrt{a}}\arctan\left(\frac{x}{\sqrt{a}}\right).$$

For the last integral, we need to assume that $a>0$. To see why the formula holds, substitute $x=\sqrt{a}u$ (work this out for yourself!).

Example 1.6. We will evaluate $\int \frac{1}{(x^2+4)^{3/2}}dx$. We see that we have a term of the form $\sqrt{x^2+4}$ (albeit cubed, but it doesn't matter), so we should make the substitution $x=\sqrt{4}\tan(\theta)=2\tan(\theta)$. So $dx=2\sec^2(\theta)d\theta$, and

$$\int \frac{1}{(x^2+4)^{3/2}}dx = \int \frac{1}{(2\sec(\theta))^3}(2\sec^2(\theta)d\theta) = \frac{1}{4}\int \frac{1}{\sec(\theta)}d\theta = \frac{1}{4}\cos(\theta)d\theta = \frac{1}{4}\sin(\arctan(x/2)).$$

1.4 Partial Fractions

The philosophy of partial fractions is that there are certain functions we should know how to integrate. These include:

1. Polynomials.
2. Functions of the form $\frac{1}{x+a}$, with antiderivative $\ln|x+a|$.
3. Functions of the form $\frac{1}{x^2+ax+b}$ (see below).
4. Functions of the form $\frac{x}{x^2+ax+b}$. Writing this as $\frac{x+a/2}{x^2+ax+b}-\frac{a/2}{x^2+ax+b}$, we find that the integral of the first term is $\frac{1}{2}\ln|x^2+ax+b|$, and the second term we know how to integrate by item (3).

Hence, for any rational function $\frac{p(x)}{q(x)}$ where p and q are polynomials, if we can express it as a combination of functions of the above forms, we will be able to integrate $\frac{p(x)}{q(x)}$. The method is as follows:

- Perform long division with p divided by q. Then we can write $\frac{p(x)}{q(x)}$ as $a(x)+\frac{b(x)}{q(x)}$, where a is a polynomial and b is the remainder when p is divided by q. So the degree of b is less than the degree of q.

- $q(x)$ can always be factored as the product $c_1(x)^{a_1}c_2(x)^{a_2}\dots c_r(x)^{a_r}d_1(x)^{b_1}d_2(x)^{b_2}\dots d_s(x)^{b_s}$, where the polynomials c_i are all *linear*, and the polynomials d_i are *irreducible quadratics* (meaning that they cannot be factored further).

- We then find the partial fraction decomposition of $\frac{b(x)}{q(x)}$. Since c_i is linear and appears a_i times in the factorization, the partial fraction decomposition includes one term of the form $\frac{A_i}{c_i(x)^k}$ for each k between 1 and a_i. Since d_i is quadratic and appears b_i times in the factorization, the partial fraction decomposition includes one term of the form $\frac{B_ix+C_i}{d_i(x)^k}$ for each k between 1 and b_i. The goal is to solve for the coefficients A_i, B_i, C_i.

- By above, we can integrate $a(x)$, each $\frac{A_i}{c_i(x)^k}$, and each $\frac{B_ix+C_i}{d_i(x)^k}$. So we can integrate $\frac{p(x)}{q(x)}$.

Example 1.7. Let's find the partial fraction decomposition of $\frac{x^3}{(x^2-1)(x^2+1)^2}$. We notice that the degree of the numerator (3) is already less than the degree of the denominator (6), so the long divison step is skipped. The denominator is not yet completely factorized: x^2-1 factors as $(x-1)(x+1)$, so the denominator has linear polynomials $x-1$ and $x+1$ each appearing once, and the irreducible quadratic x^2+1 appearing twice. Therefore the partial fraction decomposition looks like

$$\frac{x^3}{(x-1)(x+1)(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}.$$

Multiplying both sides by $x-1$ and plugging in $x=1$, we get $A = \frac{1^3}{(1+1)(1^2+1)^2} = \frac{1}{8}$. Similarly, if we multiply both sides by $x+1$ and plug in $x=-1$, we get $B = \frac{(-1)^3}{(-1-1)((-1)^2+1)^2} = \frac{1}{8}$. Now, if we multiply both sides by $(x-1)(x+1)(x^2+1)^2$ and use our previously obtained values for A and B, we get

$$x^3 = \frac{1}{8}(x+1)(x^2+1)^2 + \frac{1}{8}(x-1)(x^2+1)^2 + (Cx+D)(x-1)(x+1)(x^2+1) + (Ex+F)(x-1)(x+1).$$

If we expand the right-hand side and equate coefficients (i.e. the coefficient of x^5, x^4, x^2, x, and $x^0=1$ have to be 0, while the coefficient of x^3 is 1), we get $C=-\frac{1}{4}$, $D=0$, $E=\frac{1}{2}$, $F=0$. So the partial fraction decomposition is

$$\frac{x^3}{(x-1)(x+1)(x^2+1)^2} = \frac{1/8}{x-1} + \frac{1/8}{x+1} - \frac{x/4}{x^2+1} + \frac{x/2}{(x^2+1)^2}.$$

This is in general how a partial fraction decomposition is calculated. For *unrepeated linear factors* like $x-1$ and $x+1$ in the above Example, you can perform the "cover-up method". For *any* repeated factors, your only option is to expand both sides into polynomials and equate coefficients.

Remark 1.8. Notice that the number of coefficients you solve for should be equal to the degree of the denominator! For instance, in Example 2.7, the denominator $(x-1)(x+1)(x^2+1)^2$ has degree 6, and we are solving for 6 coefficients A through F. So if the number of coefficients does not equal the degree of the denominator, something has gone wrong.

Finally, we need to show how to integrate $\frac{1}{x^2+ax+b}$ to complete the "partial fraction heuristic".

Example 1.9. We will evaluate $\int \frac{1}{x^2+ax+b}dx$. The idea is to "complete the square" in the denominator to get rid of the ax term. Using the substitution $u - a/2 = x$ (so $du = dx$), we have

$$\int \frac{1}{x^2+ax+b}dx = \int \frac{1}{(u-a/2)^2+a(u-a/2)+b}du = \int \frac{1}{u^2-au+a^2/4+au-a^2/2+b}du$$
$$= \int \frac{1}{u^2+(-a^2/4-a^2/2+b)}du.$$

The last integral can be evaluated directly. In particular, let $c = -a^2/4 - a^2/2 + b$ for ease of notation. If $c \leq 0$, then the denominator factors as a difference of squares, and $\frac{1}{u^2+c} = \frac{1}{(u+\sqrt{-c})(u-\sqrt{-c})}$ (note that $\sqrt{-c}$ makes sense because $-c \geq 0$!), and this can be evaluated using partial fractions. If $c > 0$, then $\int \frac{1}{u^2+c}du = \frac{1}{\sqrt{c}} \arctan(u/\sqrt{c})$ (see Section 2.3).

More concretely, suppose we wish to integrate $\frac{1}{x^2+2x+4}$. Using the above method, we substitute $u - 1 = x$ and get

$$\int \frac{1}{x^2+2x+4}dx = \int \frac{1}{(u-1)^2+2(u-1)+4}du = \int \frac{1}{u^2+3}du$$
$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} \arctan\left(\frac{x+1}{\sqrt{3}}\right).$$

2 Applications of Integration

I don't have to say much about these sections. Just make sure you know the formulas.

2.1 Numerical Integration

Memorize the midpoint rule, trapezoid rule, and Simpson's rule, along with the corresponding error bounds. I won't rewrite them here, since they're in your book. *Remember that for Simpson's rule to apply, n must be even!*

A word about error bounding: in general, bounding functions is hard. But for most of the functions you will be given, there will be a simple analysis that tells you what an upper bound should be.

Example 2.1. Let's see how large n needs to be such that the trapezoid rule approximation for $\int_1^2 \ln(x)dx$ is within 1/1200 of the actual value. We know that this error E_n satisfies

$$|E_n| \leq \frac{K(2-1)^3}{12n^2} = \frac{K}{12n^2},$$

where K is an upper bound for $\left|\frac{d^2}{dx^2}\ln(x)\right| = \left|-\frac{1}{x^2}\right| = \frac{1}{x^2}$ on the interval $[1,2]$ (this means that $K \geq \frac{1}{x^2}$ for all x between 1 and 2). But we know that $\frac{1}{x^2}$ is a *decreasing* function when x is positive (when x is positive and increases, x^2 increases, so its reciprocal decreases). This implies that $\frac{1}{x^2}$ on the interval $[1,2]$ can be bounded by the value of the function at $x = 1$: $\frac{1}{1^2} = 1$, and we may therefore take $K = 1$.

So as long as $\frac{1}{12n^2} \leq \frac{1}{1200}$, we know that $|E_n|$ will be less than $\frac{1}{1200}$, as desired. Solving the inequality, we see that any $n \geq 10$ works, so the minimum possible n is 10.

2.2 Arc Length

Simply know that the length of the graph of a function $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + (f'(x))^2}dx$. Conversely, if the function is given with x in terms of y; i.e. $x = g(y)$, the length of the graph from $y = a$ to $y = b$ is $\int_a^b \sqrt{1 + (g'(y))^2}dy$.

2.3 Center of Mass

Again, just memorize the formulas; it's not necessary to know where they come from. Remember that the formulas are usually expressed in terms of $y = f(x)$; i.e. x as the independent and y as the dependent variable. But if the variables "swap roles", so we instead have $x = g(y)$, the formulas for the coordinates $(\overline{x}, \overline{y})$ of the center of mass/centroid also swap accordingly.

3 Infinite Stuff (Improper Integrals and Sequences)

3.1 Improper Integrals

To analyze improper integrals, you will need to know the comparison principle, as well as some basic convergent/divergent integrals. Here are some important reminders:

- Convergence and divergence only depends on behavior "near the asymptotes". In particular, if f is continuous on $[a, \infty)$, and $b \geq a$, then $\int_a^\infty \cdot f(x)dx$ and $\int_b^\infty \cdot f(x)dx$ have the same convergence/divergence behavior. This is because $\int_a^\infty \cdot f(x)dx = \int_a^b \cdot f(x)dx + \int_b^\infty \cdot f(x)dx$, and $\int_a^b \cdot f(x)dx$ is just some finite number.

- Convergence and divergence also does not depend on any nonzero constants that are introduced. This means that if $c \neq 0$ is a constant, then $\int_1^\infty c \cdot f(x)dx$ has the same convergence/divergence behavior as $\int_1^\infty \cdot f(x)dx$

- For double-sided improper integrals of the form $\int_{-\infty}^\infty f(x)dx$, **never "plug in infinity and minus infinity"**; this is meaningless and is very likely to give you nonsensical (i.e. wrong) results. Instead, write the integral as $\int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$ (assuming f has no other asymptotes) and analyze the two pieces separately: $\int_{-\infty}^\infty f(x)dx$ converges means that both $\int_{-\infty}^0 f(x)dx$ and $\int_0^\infty f(x)dx$ converge.

- Similarly, if we have an improper integral of the form $\int_a^b f(x)dx$ where f has an asymptote at some c between a and b, we should split the integral up as $\int_c^b f(x)dx + \int_a^c f(x)dx$ and analyze the two parts separately.

Proposition 3.1 (Comparison Principle)**.** Suppose f and g are functions such that $0 \leq f(x) \leq g(x)$ for all $x \geq a$, where a is some constant. Then if $\int_a^\infty g(x)$ converges, so does $\int_a^\infty f(x)dx$. If $\int_a^\infty f(x)dx$ diverges, then so does $\int_a^\infty g(x)dx$.

It is important to note that f and g must be nonnegative! To see why this is necessary, consider $f(x) = -x$ and $g(x) = 0$. Then $f(x) \leq g(x)$ for all $x \geq 0$, and $\int_0^\infty g(x)dx$ converges and equals 0, but clearly $\int_0^\infty f(x)dx$ diverges.

The question remains as to how to find the proper comparison to decide convergence/divergence of an improper integral. Here are essentially all the facts you need to know:

Fact 3.2. $\int_1^\infty \frac{1}{x^p}dx$ converges exactly when $p > 1$, and diverges otherwise. $\int_0^1 \frac{1}{x^p}dx$ converges exactly when $p < 1$, and diverges otherwise.

Sometimes it is useful to use a more general result: $\int_2^\infty x^a(\ln x)^b dx$ converges if $a < -1$, or if $a = -1$ and $b < -1$. It diverges in all other cases. We start the integral at the lower bound 2 (instead of 1) so the integrand has no other asymptotes.

Fact 3.3 (The "hierarchy of functions")**.** There is a general hierarchy of functions in relation to their growths at infinity. For continuous functions on $[1, \infty)$, it goes:

1. Bounded functions (e.g. constants, $\sin(x)$, e^{-x} on $[1, \infty)$) grow the slowest.
2. Logarithmic functions (e.g. $\ln(x), \ln(3x^4 + 2x^2)$) tend to ∞, unlike bounded functions.
3. Power functions of the form $x^a = e^{a \ln x}$ for any $a > 0$ grow faster than any logarithmic function.
4. Exponential functions (e.g. e^x, or in general any $e^{f(x)}$ where f is a power function) grow faster than any power function.

Each class of functions grows so much faster than classes of functions lower on this hierarchy. As an example:

Example 3.4. We will determine the behavior of

$$\int_1^\infty \frac{x^3 \ln(x)}{(x^5 + 1)(\sin^2(x^2 + \cos(2x + 4)) + 2)} dx.$$

Notice that $\sin^2$ of anything is always between 0 and 1, so the $\sin^2(x^2 + \cos(2x + 4)) + 2$ is between 2 and 3. Also, $x^5 + 1 > x^5$ for any $x \geq 1$, so we can make the comparison

$$\int_1^\infty \frac{x^3 \ln(x)}{(x^5 + 1)(\sin^2(x^2 + \cos(2x + 4)) + 2)} dx \leq \int_1^\infty \frac{x^3 \ln(x)}{(x^5)(2)} dx = \frac{1}{2} \int_1^\infty \frac{\ln(x)}{x^2} dx.$$

Therefore by comparison, if $\int_1^\infty \frac{\ln(x)}{x^2} dx$ converges, then so does our original integral. But the hierarchy of functions tells us that $\ln(x)$ grows slower than x^a for any $a > 0$. So in the long run, we have $\ln(x) < x^{1/2}$, and we can write (using the $\ll$ notation to make it precise, but it doesn't matter what this means for Math 1B purposes):

$$\int_1^\infty \frac{\ln(x)}{x^2} dx \ll \int_1^\infty \frac{x^{1/2}}{x^2} dx = \int_1^\infty \frac{1}{x^{3/2}} dx.$$

As $3/2 > 1$, this last integral converges (see Fact 4.2), so our original integral converges as well.

Personally, I like to make various substitutions to get the bounds of integration from a to ∞, because the hierarchy of functions only applies for the behavior of functions at infinity (do *not* try to apply it for function behavior on $[0, 1]$ or any other interval!) Here is an example of what I mean:

Example 3.5. We will determine the behavior of

$$\int_1^2 \frac{1}{(x-1)^{5/2}e^{1/(x-1)}}dx.$$

Now, we do not have any facts about integrating over this non-standard interval $[1,2]$. But via some u-substitutions, we can transform this to something we are more familiar with. With $u = x-1$, we have

$$\int_1^2 \frac{1}{(x-1)^{5/2}e^{1/(x-1)}}dx = \int_0^1 \frac{1}{u^{5/2}e^{1/u}}du.$$

Now set $v = \frac{1}{u}$, so $du = -\frac{1}{v^2}dv$, and we get

$$\int_0^1 \frac{1}{u^{5/2}e^{1/u}}du = \int_\infty^1 -\frac{1}{(1/v)^{5/2}e^v v^2}dv = \int_1^\infty \frac{v^{1/2}}{e^v}dv.$$

Then by the hierarchy of functions, because the exponential function e^v grows so much faster than the power function $v^{1/2}$, this final integral converges (essentially since $\int_1^\infty \frac{1}{e^v}dv$ converges). So the original integral converges as well.

3.2 Sequences

You should know the *rigorous* definition of the limit of a sequence:

Definition 3.6. A sequence $\{a_n\}$ converges to a limit L if, for every $\epsilon > 0$, there is a positive integer N such that whenever $n \geq N$, then $|a_n - L| < \epsilon$.

To determine convergence or divergence of sequences, the same principles that we used in the analysis of improper integrals apply. These include the hierarchy of functions and the comparison principle.

Example 3.7. The sequence $a_n = \frac{n^{100000000000}}{e^n}$ converges to 0 as $n \to \infty$. Indeed, by the hierarchy of functions, e^n grows much faster in the long run than any power function, so the limit is 0.

We also have the squeeze rule:

Proposition 3.8. Suppose we have sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$ such that $a_n \leq b_n \leq c_n$ for all n. If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} c_n$ both converge to the same limit L, then $\lim_{n\to\infty} b_n = L$.

Example 3.9. The sequence $a_n = \frac{\sin(n)}{n^2}$ converges to 0 as $n \to \infty$. We can see this in 2 ways. One is by the hierarchy of functions, using that $\sin(n)$ is bounded and n^2 grows to infinity. We can also use the squeeze rule, which is logically the same: since $-1 \leq \sin(n) \leq 1$ for all n, we know that

$$-\frac{1}{n^2} \leq a_n = \frac{\sin(n)}{n^2} \leq \frac{1}{n^2},$$

and since both $-\frac{1}{n^2}$ and $\frac{1}{n^2}$ converge to 0 as $n \to \infty$, the squeeze rule says that $\lim_{n\to\infty} a_n = 0$ as well.

The monotone convergence theorem is useful for determining that the limit of a sequence exists, but not what the limit actually is.

Proposition 3.10. Suppose $\{a_n\}$ is a sequence that monotonically increases. Then if $\{a_n\}$ is bounded above by some integer M, by which we mean $a_n \leq M$ for all n, then $\{a_n\}$ converges. Note that we do not claim that the limit is M.

Similarly, if $\{a_n\}$ monotonically *decreases* and is bounded *below*, then it converges as well.

Finally, we may be in a situation where the sequence is defined *recursively*; i.e. where each term a_n is defined via previous terms a_{n-1}, a_{n-2}, etc. (and the initial term a_1 should be given to you). The limits of such sequences can be found with the following fact:

Proposition 3.11. If a_n converges, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

This can be proved by going back to Definition 4.6. Intuitively, the idea is that if a_n is "arbitrarily close to L in the long run", then so will a_{n+1} (since $\{a_{n+1}\}$ is just a re-indexing of the sequence $\{a_n\}$—the same numbers in the same order but labeled differently).

Example 3.12. Consider the sequence $\{a_n\}$ given by $a_1 = 3$, $a_{n+1} = \sqrt{2a_n + 8}$. We will show that $\{a_n\}$ converges and find the corresponding limit.

The strategy for this problem is a bit of "backwards reasoning". Let's first *assume* that $\{a_n\}$ converges. Then we can apply Proposition 4.11: writing L for $\lim_{n\to\infty} a_n$, we have

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n + 8} = \sqrt{2\left(\lim_{n\to\infty} a_n\right) + 8} = \sqrt{2L + 8}.$$

So $L^2 = 2L+8$, and solving the quadratic, we get $L = -2$ or $L = 4$. But -2 is an extraneous solution (since $\sqrt{2(-2)+8}$ is 2, not -2), so $L = 4$ is the only possibility. Therefore *if* we show that $\{a_n\}$ converges, then the limit must be 4.

To show that $\{a_n\}$ converges, we will use the monotone convergence theorem, so we want to show that $\{a_n\}$ is increasing and bounded above. Notice that all of the a_n are positive, because $a_0 = 3$ is positive, and if x is positive then so is $\sqrt{2x+8}$. Now, notice that if

$1 \leq x \leq 4$, then $10 \leq 2x + 8 \leq 16$, so $\sqrt{10} \leq \sqrt{2x+8} \leq 4$. The upshot is that since $a_0 = 3$ is between 1 and 4, then $a_1 = \sqrt{2a_0 + 8}$ is between 1 and 4 (since $\sqrt{10} > 1$), so that $a_2 = \sqrt{2a_1 + 8}$ is between 1 and 4, and so on. Using this logic, it follows that all of the a_n are between 1 and 4 (this is an example of *mathematical induction*). In particular the $\{a_n\}$ are bounded, which is half of what we want.

It remains to show that $\{a_n\}$ is increasing, so we need to show that $a_n \leq a_{n+1} = \sqrt{2a_n + 8}$. Both sides are positive, so we can square both sides and show that $a_n^2 \leq 2a_n + 8$, or that $a_n - 2a_n - 8 \leq 0$. Since the left side factors as $(a_n - 4)(a_n + 2)$, and *above we saw that the a_n are all between 1 and 4*, $(a_n - 4)(a_n + 2)$ is the product of something nonpositive and something positive, implying that $a_n - 2a_n - 8 = (a_n - 4)(a_n + 2) \leq 0$. So we get that $a_n \leq a_{n+1}$.

The conclusion is that the a_n are increasing and bounded above, so the sequence $\{a_n\}$ converges. But remember that we showed that *if* $\{a_n\}$ converges, then its limit is 4!

Warning: be careful about the assumptions that you make involving limits. For instance, the method shown in the above Example 4.12 only works *if you already know, or if you show, that $\{a_n\}$ converges.* Otherwise you will get nonsensical answers. As an example of such an absurdity, suppose $a_1 = 1$ and $a_{n+1} = -a_n$. Then by a bogus application of Proposition 4.11, we get

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (-a_n) = -L,$$

so we "conclude" that $L = 0$. But the sequence $\{a_n\}$ is $\{1, -1, 1, -1, \ldots\}$, which obviously doesn't converge.

Similarly, it is *incorrect* to use the equation

$$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n + b_n)$$

if you don't know that $\{a_n\}$ and $\{b_n\}$ actually converge! As an example, consider the sequences $a_n = n + 1$, $b_n = -n$. Then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} (n + 1 - n) = \lim_{n\to\infty} 1 = 1,$$

but $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$. Then the "equation" $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n + b_n)$ becomes the nonsensical thing $\infty - \infty = 1$.

Therefore, the correct thing to do is to *first* show that sequences you are dealing with converge, and only then apply various limit laws to find the limit.